VSE Econ502 Advanced Macroeconomics Notes , Nov 2018

Continuous Time Optimization: Hamiltonians

Tutorial Notes

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1 General Cookbook

General Set-up: The deterministic optimal control problem can be generally described with following parts:

- Vector of state variable: $x_t \in X$; vector of control variable: $z_t \in Z$, and both are functions of time t. Note the difference between choice and state variables is that state variables are predetermined thus they are usually captured by law of motions, whereas choice variables are optimally chosen in each period.
- Instantaneous pay-off function: $h(x_t, z_t) : X \times Z \to R$
- Discount rate: $\rho > 0$

The optimal control problem is then formed as:

$$\max_{z_t, x_t} \int_0^{\infty} e^{-\rho t} h(x_t, z_t) dt$$

$$s.t. \quad \frac{dx_t}{dt} \equiv \dot{x}_t = g(x_t, z_t)$$

Where $g(x_t, z_t)$ is a control function that describes the law of motion of state vector.

Formulation: 1. **Present-value Hamiltonian** (similar to its discrete time counter-part of Lagrangian):

$$H(x_t, z_t, \lambda_t) = e^{-\rho t} h(x_t, z_t) + \lambda_t g(x_t, z_t)$$

Take first order conditions (FOC):

- Control variable: z_t

$$H_z(x_t, z_t, \lambda_t) = 0$$

- State variable: x_t

$$\dot{\lambda}_t = -H_x(x_t, z_t, \lambda_t)$$

- Law of motion:

$$\dot{x}_t = g(x_t, z_t)$$

2. **Current-value Hamiltonian** (similar to its discrete time counter-part of Lagrangian):

$$\hat{H}(x_t, z_t, \mu_t) = h(x_t, z_t) + \mu_t g(x_t, z_t)$$

Take first order conditions (FOC):

- Control variable: z_t
$$\hat{H}_z(x_t, z_t, \mu_t) = 0$$
- State variable: x_t
$$\dot{\mu}_t = \rho\mu_t - \hat{H}_x(x_t, z_t, \mu_t)$$
- Law of motion:
$$\dot{x}_t = g(x_t, z_t)$$

For the infinite period problem to be well-defined, we need two boundary conditions:

(1). Initial condtion:

$$x_0 = \hat{x}_0$$

(2). Transversality Condition (TVC):

$$\lim_{T\to\infty} e^{-\rho T}\lambda_T x_T$$

or

$$\lim_{T\to\infty} \mu_T x_T$$

2 An example: Neoclassical Growth Model

Set-up:

- State variable: capital $k_t \in X$; control variable: consumption $c_t \in Z$, and both are functions of time t.
- Instantaneous pay-off function: $u(c_t)$
- Discount rate: $\rho > 0$
- Law of motion: $\dot{k}_t = f(k_t) - \delta k_t - c_t$

Formulation of Present-Value Hamiltonian: The optimal control problem then can be formed as:

$$\max_{c_t, k_t} \int_0^\infty e^{-\rho t} u(c_t)$$
$$s.t. \quad \dot{k}_t = f(k_t) - \delta k_t - c_t$$

Form Hamiltonian:

$$H(k, c, \lambda) = e^{-\rho t} u(c_t) + \lambda_t (f(k_t) - \delta k_t - c_t)$$

FOC:

$$[c_t]: \quad e^{-\rho t} u'(c_t) = \lambda_t \tag{1}$$

$$[k_t]: \quad \dot{\lambda}_t = -\lambda_t (f'(k_t) - \delta) = \lambda_t (\delta - f'(k_t)) \tag{2}$$

$$[\lambda_t]: \quad \dot{k}_t = f(k_t) - \delta k_t - c_t \tag{3}$$

Euler Equation: To obtain the Euler Equation, take logs on both sides of equation (1) and take derivatives w.r.t t get:

$$\frac{u''(c_t)}{u'(c_t)} \dot{c}_t - \rho = \frac{\dot{\lambda}_t}{\lambda_t}$$

Plug into (2) we get:

$$\frac{u''(c_t)}{u'(c_t)} \dot{c}_t = \rho + \delta - f'(k_t) \tag{4}$$

Formulation of Current-Value Hamiltonian:

$$\hat{H}(k, c, \mu) = u(c_t) + \mu_t (f(k_t) - \delta k_t - c_t)$$

FOC:

$$[c_t]: \quad u'(c_t) = \mu_t \tag{5}$$

$$[k_t]: \quad \dot{\mu}_t = \rho \mu_t - \mu_t (f'(k_t) - \delta) = \mu_t (\rho + \delta - f'(k_t)) \tag{6}$$

$$[\mu_t]: \quad \dot{k}_t = f(k_t) - \delta k_t - c_t \tag{7}$$

Euler Equation: To obtain the Euler Equation, take logs on both sides of equation (5) and get:

$$\frac{u''(c_t)}{u'(c_t)} \dot{c}_t = \frac{\dot{\mu}_t}{\mu_t}$$

Plug into (2) we get:

$$\frac{u''(c_t)}{u'(c_t)} \dot{c}_t = \rho + \delta - f'(k_t) \tag{8}$$

From (6) and (2) we see the difference between current and present value Hamiltonian is defined through following equation:

$$\frac{\dot{\mu}_t}{\mu_t} = \rho + \frac{\dot{\lambda}_t}{\lambda_t}$$

2.1 A Detour: Comparison with Discrete Time Method

In this section I use the familiar discrete time set-up to shed some lights on its continuous counter-part as we did before, the argument in this section is for purpose of illustration, thus I'm skipping potential issues such as measurability etc.

Consider the discrete time set up of growth model, where $\Delta > 0$ is an extremely small increment of time:

$$\max_{c_t, k_t+\Delta} \sum_t \beta^t u(c_t)$$
$$s.t. \quad \frac{k_{t+\Delta} - k_t}{\Delta} = f(k_t) - \delta k_t - c_t$$

We can then form our Lagrangian as:

$$L = \sum_t \beta^t [u(c_t) + \mu_t((f(k_t) - \delta k_t - c_t)\Delta - (k_{t+\Delta} - k_t))]$$

Now we need to choose c_t and $k_{t+\Delta}$ for each $\Delta > 0$, FOC will be:

$$[c_t]: \quad u'(c_t) = \mu_t \tag{9}$$
$$[k_{t+\Delta}]: \quad -\beta^t \mu_t + \beta^{t+\Delta} \mu_{t+\Delta}[f'(k_{t+\Delta})\Delta + 1 - \delta\Delta] = 0 \tag{10}$$
$$[\mu_t]: \quad \frac{k_{t+\Delta} - k_t}{\Delta} = f(k_t) - \delta k_t - c_t \tag{11}$$

Now rearrange (6) we get:

$$\beta^\Delta(\mu_{t+\Delta} - \mu_t) = \mu_t(1 - \beta^\Delta) - \beta^\Delta \mu_{t+\Delta}\Delta(f'(k_{t+\Delta}) - \delta)$$
$$\Rightarrow \frac{\mu_{t+\Delta} - \mu_t}{\Delta} = \mu_t \frac{1 - \beta^\Delta}{\beta^\Delta \Delta} - \beta^\Delta \mu_{t+\Delta}(f'(k_{t+\Delta}) - \delta)$$

Now let $\beta = e^{-\rho}$ as we have in continuous time, and take limit on both sides: $\Delta \to 0$

$$LHS = \lim_{\Delta \to 0} \frac{\mu_{t+\Delta} - \mu_t}{\Delta} = \dot{\mu}_t$$
$$\lim_{\Delta \to 0} \frac{1 - \beta^\Delta}{\beta^\Delta \Delta} = \lim_{\Delta \to 0} \frac{\rho}{\beta^\Delta - \rho\Delta} = \rho$$

Where the second equality follows from L'Hospital Rule. Then we have:

$$\dot{\mu}_t = \mu_t \rho - \mu_t(f'(k_t) - \delta)$$

Which is exactly the same as (2); and for (7), takes limit on both side gives equation (3).

And it's also easy to show the general formulation of Hamiltonian is consistent with Lagrangian following the similar rationale.